# Symmetry in Optimal Control

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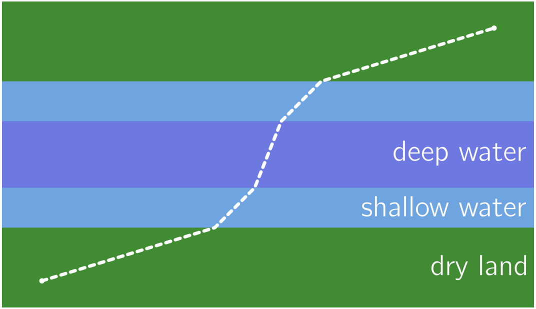
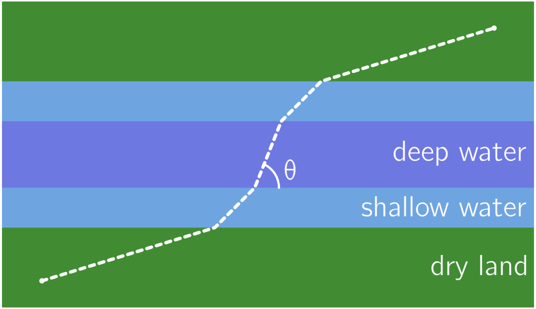
- Optimization problem: crossing a river.



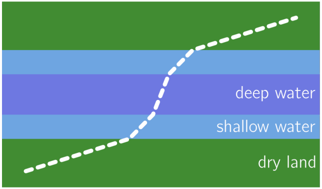


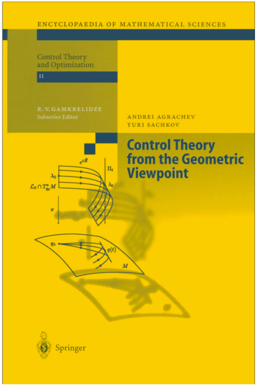

- Solution (Snell's law): over an optimal path,

$$\frac{\cos(\theta)}{V} = \text{constant}.$$

- Claim: this is implied by the problem's *horizontal symmetry*.

Let's consider the perspective of optimal control. . .



# Optimal Control

- A **control system** on $M$ is a family $(X_u)_{u \in U}$ of vector fields.
- We want a control function $u\colon [0, T] \to U$ solving

$$\begin{aligned} &\text{maximize } F(q(T)) \\ &\text{subject to } \begin{cases} q(0) = q_0 \\ \dot{q}(t) = X_{u(t)}(q). \end{cases} \end{aligned}$$

- This is the **Mayer problem** of optimal control.

# Pontryagin Maximum Principle

- Given a self-diffeomorphism $Q\colon M \to M$, define the pullback

$$Q^*\colon T^*M \to T^*M$$
$$Q^*(p) = (dQ(\pi(p))^*)^{-1}(p)$$

- The **Hamiltonian lift** is the derivative of the pullback operation:

$$\begin{array}{ccc} \mathrm{Aut}(M) & \xrightarrow{\text{pullback}} & \mathrm{Aut}(T^*M) \\ \text{exp}\uparrow & & \uparrow\text{exp} \\ D(M) & \xrightarrow{\text{Hamiltonian lift}} & D(T^*M) \end{array}$$

- If $\phi^t$ is the flow of $X$, then $(\phi^t)^*$ is the flow of $\overline{X}$.
- In coordinates,

$$\overline{X} = X_i \frac{\partial}{\partial q_i} - p_j \frac{\partial X_j}{\partial q_i} \frac{\partial}{\partial p_i}.$$

# Pontryagin Maximum Principle

## Theorem (PMP)

*Suppose* $q\colon [0,T] \to M$ *is a solution to the Mayer problem*

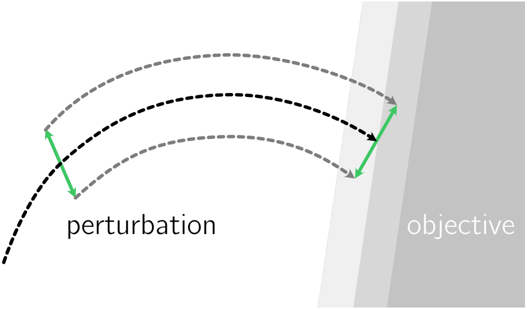
$$\text{maximize } F(q(T))$$
$$\text{subject to } \begin{cases} q(0) = q_0 \\ \dot{q} = X_u(q). \end{cases}$$

*Then there exists a path* $\lambda\colon [0,T] \to T^*M$ *so that* $\pi \circ \lambda = q$ *and the following conditions hold:*

1. **The adjoint system:** $\dot{\lambda} = \overline{X_u}(\lambda)$.
2. **Maximality:** $\dot{q} = X_u(q)$ *maximizes* $\langle p, \dot{q} \rangle$.
3. **Transversality:** $\lambda(T) = dF(q(T))$.

# Pontryagin Maximum Principle

- Suppose the control $u$ is modified at a particular moment $t$. Then the trajectory is perturbed, affecting $F(q(T))$.



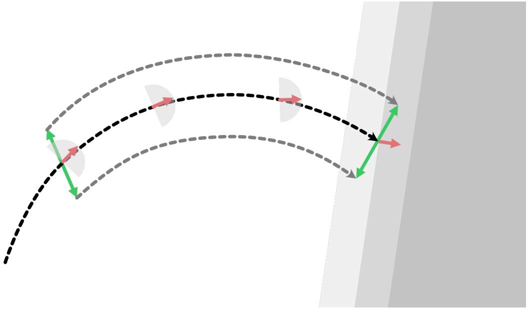

- First-order optimality condition: $u$ is chosen so that $\langle dF(q(T)), d\phi_t^T(\dot{q})\rangle$ is maximized.

# Pontryagin Maximum Principle

- Rewrite in terms of pullback of $dF(q(T))$:

$$\langle dF(q(T)), d\phi_t^T(\dot{q})\rangle = \langle (\phi_T^t)_*(dF(q(T))), \dot{q}\rangle$$

- Let $\lambda(t) = (\phi_T^t)_*(dF(q(T)))$.



Then $\dot{\lambda} = \overline{X_u}(\lambda)$.

# Pontryagin Maximum Principle

## Definition

A curve $(p, q) = \lambda \in T^*M$ is **extremal** when

$$\dot{\lambda} = \overline{X_u}(\lambda), \quad \langle p, \dot{q} \rangle = \max_{u \in U} \langle p, X_u(\lambda) \rangle$$

at (almost) all instants.

- Slogan: solutions to optimal control problems are projections of extremal curves.

# Pontryagin Maximum Principle

- The Pontryagin maximum principle seems to involve the vector fields $\overline{X_u}$...
- ...but our control problem only depends on the sets

$$V(q) = \{X_u(q) : u \in U\}.$$

- Two families $X_u$ and $Y_u$ giving the same sets $V(q)$ are called *feedback*- or *gauge-equivalent*.
- Can we describe extremal curves in terms of the sets $V(q)$?

# The Maximized Hamiltonian

## Definition

The **maximized Hamiltonian** $H\colon T^*M \to \mathbb{R}$ is defined by

$$H(q,p) = \sup_{v \in V(q)} \langle p, v \rangle.$$

## Theorem (Well-known)

*If $H$ is everywhere differentiable and $\lambda$ is extremal, then*

$$\dot{\lambda} = (dH(\lambda))^{\#}$$

*almost everywhere.*

# The Maximized Hamiltonian

## Theorem (Generalization)

*If $\lambda$ is extremal, then*

$$\dot{\lambda} \in \partial_C H(\lambda)^{\#}$$

*almost everywhere.*

## Proof.

Consider an instant $t$ at which

$$\begin{cases} \dot{\lambda} = \overline{X_u}(\lambda) \\ \langle p, X_u(q) \rangle = \sup_{u^* \in U} \langle p, X_{u^*}(q) \rangle = H(\lambda). \end{cases}$$

Define $H_u = \langle -, X_u \rangle$. Then $H_u(\lambda) = H(\lambda)$ and $H_u \leq H$ everywhere. So $dH_u(\lambda) \in \partial_C H(\lambda)$, and

$$\dot{\lambda} = \overline{X_u}(\lambda) = dH_u(\lambda)^{\#} \in \partial_C H(\lambda)^{\#}.$$

□

# The Maximized Hamiltonian

Corollary

*The maximized Hamiltonian $H$ is conserved over extremal curves.*

Corollary

*If $Y$ is a vector field that respects velocity sets, meaning*

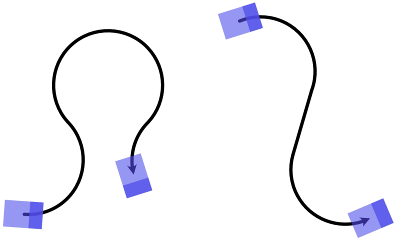
$$\forall t, de^{tY}(V(p)) = V(e^{tY}(p)),$$

*then $H_Y = \langle Y, - \rangle$ is conserved over extremal curves.*

- Similar to *Conservation Laws in Optimal Control*, Delfim Torres, 2002.

# Example: Curves of Bounded Curvature

- What are the shortest curves of **bounded curvature** with prescribed initial and final directions?
- In two dimensions: optimal control of a car.



# Example: Curves of Bounded Curvature

- Let $M = \mathbb{R}^n \times \mathbb{R}^n$ and

$$V(x, \theta) = \{(\theta, v) : \langle v, \theta \rangle = 0, \|v\| = 1\}.$$

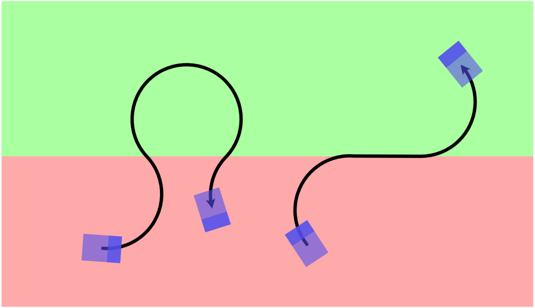
- Let $(p_x, p_\theta)$ be the costate variables. For an extremal curve,

$$\begin{cases} \dot{x} = \theta \\ \dot{\theta} = \underset{\{v : \langle v, \theta \rangle = 0\}}{\operatorname{argmax}} \langle p_\theta, v \rangle \end{cases} \quad \begin{cases} \dot{p}_x = 0 \\ \dot{p}_\theta = \dots \end{cases}$$

- By **translational symmetry**, $p_x$ is constant.
- By **rotational symmetry**, $p_x \wedge x + p_\theta \wedge \theta$ is constant.

# Curves of Bounded Curvature

- In two dimensions, $p_\theta \wedge \theta = -p_x \wedge x + C$ has a simple interpretation.



# Further Generalizations

- What about the Mayers problem

$$\text{maximize } F(q(T))$$
$$\text{subject to } \begin{cases} q(0) = q_0 \\ \dot{q}(t) \in V(q(t))? \end{cases}$$

- The maximized Hamiltonian may be *discontinuous*...
- Is there still a first-order optimality theory?

# Further Generalizations

- The PMP can be derived from a Lagrange multipliers approach, using the Lagrangian function

$$\mathcal{L}(q,p) = F(q(T)) + \int_0^T \langle p, \dot{q} - X_u \rangle \, dt.$$

- It is natural to pass to the *Hamiltonian action*

$$\mathcal{L}(q,p) = F(q(T)) + \int_0^T \langle p, \dot{q} \rangle - H(q,p) \, dt.$$

- *Generalized Hamiltonian Mechamics*, J. E. Marsden, 1968
- The dynamical laws governing extremal trajectories are more complicated, but **Noether's theorem is preserved**.

# Example: Crossing a River

- Let $V(x, y) = \{\dot{q} : \|\dot{q}\| \leq v(y)\}$.
- Then $H(x, y, p_x, p_y) = v(y)\|(p_x, p_y)\|$.
- Since $H$ and $p_x$ are conserved quantities,

$$\frac{p_x}{H} = \frac{p_x}{v(y)\|(p_x, p_y)\|} = \frac{\cos(\theta)}{v(y)}$$

is conserved along an extremal curve and hence along an optimal trajectory.