Theorem 1. *The maps*

$$D^*(M) \xrightarrow{H_{(-)}} C^\infty T^*M \xrightarrow{\overrightarrow{(-)}} D(T^*M)$$

are Lie algebra homomorphisms.

Proof. That the map $\overrightarrow{(-)}$ is a Lie algebra homomorphism was already proved in the thesis. (Its homomorphism law follows from the Jacobi identity.) We proceed to the case of the map $H_{(-)}$.

Let $Y \in D(M)$ and $Q \in \mathrm{Aut}(M)$. We claim first that

$$H_{(\operatorname{Ad} Q)Y} = (T^*Q)H_Y.$$

Indeed, evaluating these elements of $C^\infty T^*M$ at an arbitrary point λ gives

$$\begin{aligned} H_{(\operatorname{Ad} Q)Y}(\lambda) &= \langle \lambda, QYQ^{-1} \rangle \\ &= \langle (dQ^{-1}_{Q(\pi(\lambda))})^*(\lambda), Y \rangle = \lambda(T^*Q)(H_Y). \end{aligned}$$

Next, let $X \in D^*(M)$. Applying the previous formula together with Lemma 3 from our thesis, we derive that

$$\begin{aligned} H_{[X,Y]} &= \left(\frac{d}{dt}\right)_{t=0} H_{(\operatorname{Ad}(\exp(tX)))Y} = \left(\frac{d}{dt}\right)_{t=0} (T^* \exp(tX))(H_Y) \\ &= \overline{X}(H_Y) = \{H_X, H_Y\}. \end{aligned}$$

Of course, the map $H_{(-)}$ is also linear, so we conclude that it is a Lie algebra homomorphism. □

Remark 1. The identity $H_{[X,Y]} = \{H_X, H_Y\}$ can also be checked in coordinates, which shows (unsurprisingly) that the theorem above is true if we replace $D^*(M)$ with $D(M)$.

Theorem 2 (Noether's Theorem for the PMP)**.** *Let (M, U, X) be a parameterized control system, and suppose $Y \in D^*(T^*M)$ is such that*

$$V(p)\exp(tY) = V(p\exp(tY))$$

for all $p \in M$ and $t \in \mathbb{R}$. Then H_Y is constant along any extremal trajectory.

Proof. Let λ be an extremal trajectory driven by a control function u. For almost any t, the following two conditions hold:

1. $\dot{\lambda}(t) = \lambda(t)\overline{X_{u(t)}}$.
2. $\langle \lambda(t), X_w \rangle$ is maximized in the variable $w \in U$ by $w = u(t)$.

Let t be a fixed instant in these conditions and abbreviate

$$\lambda = \lambda(t), \quad p = p(t) = \lambda(t)\pi, \quad X = X(t).$$

We will prove that

$$\frac{d}{dt}\lambda(t)(H_Y) = \lambda\overline{X}(H_Y) = 0.$$

Consider the expression

$$\alpha(s) = \langle \lambda, (\operatorname{Ad}\exp(sY))X \rangle.$$

By hypothesis on Y, $p(\operatorname{Ad}\exp(sY))X$ must belong to $V(p)$ for all $s \in \mathbb{R}$. It follows from condition (2) above that $\alpha(s)$ attains a maximum at $s = 0$. Since α is differentiable, we must have $\dot{\alpha}(0) = 0$. On the other hand, expanding $\dot{\alpha}(0)$ with the help of Theorem 1 shows

$$\dot{\alpha}(0) = \langle \lambda, [Y, X] \rangle = \lambda H_{[Y,X]} = \lambda\{H_Y, H_X\}.$$

We conclude, as desired, that

$$\lambda\overline{X}(H_Y) = \lambda\{H_X, H_Y\} = -\lambda\{H_Y, H_X\} = 0.$$

□